Complexity of Teaching by a Restricted Number of Examples

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Background

• Computational **teaching** theory
  – Aims to bring out the nature of teaching
  – which is inextricably linked to learning
    • Teachability [Shinohara and Miyano 1991]
    • Teaching dimension [Goldman and Kearns 1991]
  ...
    • Expected teaching dimension [Balbach 2005]
    • Recursive teaching dimension [Zilles et al. 2008]
Illustrative problem: Phone-a-friend lifeline

• **Millionaire** (=“Who Wants to Be a Millionaire?”)
  – Challenge multiple-choice questions
  – Win a cash award depending on the number of correct answers
  – Get help from the three lifelines during the game

• **Lifelines**
  – **Phone-a-friend**
    • Will give you advice from friends
  – **50:50**
    • Removes two incorrect answers
  – **Ask the Audience**
    • Lets you see the answers of audience
Classical model (1/3)

• Millionaire (one correct choice)

When was the COLT conference first held?

A: 1983
B: 1988
C: 1992
D: 1997

Concept class (available answers)
\( C = \{\{A\}, \{B\}, \{C\}, \{D\}\} \)
Classical model (1/3)

• Millionaire (one correct choice)

When was the COLT conference first held?

- A: 1983
- B: 1988
- C: 1992
- D: 1997

Concept class (available answers)

\[ C = \{\{A\}, \{B\}, \{C\}, \{D\}\} \]

Target concept (correct answer)

\[ c = \{B\} \]

Learner (challenger)

Teaching

Teacher (friend)
Classical model (1/3)

• Millionaire (one correct choice)

When was the COLT conference first held?

A: 1983

B: 1988

C: 1992

D: 1997

Concept class (available answers)
\( C = \{\{A\}, \{B\}, \{C\}, \{D\}\} \)

Target concept (correct answer)
\( c = \{B\} \)

Teaching set
\( S = \{(A, \text{False}), (B, \text{True}), (C, \text{False}), (D, \text{False})\} \)

Learner (challenger)

Teaching

Teacher (friend)
Classical model (1/3)

• Millionaire (one correct choice)

When was the COLT conference first held?

A: 1983

B: 1988

C: 1992

Concept class (available answers)

\[ C = \{ \{A\}, \{B\}, \{C\}, \{D\} \} \]

Teaching set

\[ S = \{(A, \text{False}), (B, \text{True}), (C, \text{False}), (D, \text{False})\} \]

Teaching Dimension

\[ TD(c, C) := |\text{Minimum teaching set}| \]

In this case, \( TD(c, C) = 1 \)

\[ c = \{B\} \]

Teacher (friend)

Learner (challenger)
Classical model (2/3)

• Millionaire 2.0 (two correct choices)

Which are the two cities where the Olympic games were held in Canada?

A: Montreal
B: Calgary
C: Ottawa
D: Vancouver

Concept class (available answers)

\[ C = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \]
Classical model (2/3)

- Millionaire 2.0 (two correct choices)

Which are the two cities where the Olympic games were held in Canada?

A: Montreal
B: Calgary
C: Ottawa
D: Vancouver

Concept class (available answers)
\( C = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \)

Target concept (correct answer)
\( c = \{A, B\} \)

\( S = \{(A, \text{True}), (B, \text{True})\} \)

Learner (challenger)

Teaching

Teacher (friend)
Classical model (2/3)

• Millionaire 2.0 (two correct choices)

Which are the two cities where the Olympic games were held in Canada?

A: Montreal
B: Calgary
C: Ottawa

Concept class (available answers)

\[ C = \{\{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \{C,D\}\} \]

Minimum teaching set

\[ S = \{\langle A, \text{True} \rangle, \langle B, \text{True} \rangle\} \]

Teaching Dimension

\[ \text{TD}(c, C) := |\text{Minimum teaching set}| \]

In this case, \( \text{TD}(c, C) = 2 \)

\[ c = \{A, B\} \]
Classical model (3/3)

• Generalized Millionaire (unknown # of correct choices)

Which choices are correct specialties of Canada?

A: Maple tea
B: Maple butter
C: Maple dressing
D: Maple mustard

Concept class (available answers) \( C = 2^{\{A,B,C,D\}} \)
Target concept (correct answer) \( c = \{A, B, C, D\} \)

S = {(A, True), (B, True), (C, True), (D, True)}

Learner (challenger)
Teacher (friend)
Classical model (3/3)

• **Generalized Millionaire** (unknown # of correct choices)

Which choices are correct specialties of Canada?

- A: Maple tea
- B: Maple butter
- C: Maple dressing
- D: Maple mustard

Concept class (available answers)  
\[ C = 2^{\{A, B, C, D\}} \]

Target concept (correct answer)  
\[ c = \{A, B, C, D\} \]

Learner (challenger)  
Teacher (friend)
Classical model (3/3)

• **Generalized Millionaire** (unknown # of correct choices)

Concept class (available answers)  
\[ C = 2^{\{A, B, C, D\}} \]

Target concept (correct answer)  
\[ c = \{A, B, C, D\} \]

Which choices are correct specialties of Canada?

A: Maple tea  
B: Maple butter  
C: Maple dressing  
D: Maple mustard

Minimum teaching set  
\[ S = \{(A, \text{True}), (B, \text{True}), (C, \text{True}), (D, \text{True})\} \]

Teaching Dimension  
\[ \text{TD}(c, C) := |\text{Minimum teaching set}| \]

In this case, \( \text{TD}(c, C) = 4 \)
Our contributions

• **When # of examples < teaching dimension**

• Formal proofs that
  – Special teaching strategies are necessary
    • A subset of a teaching set is not always optimal
  – Smart teachers dare to tell a lie
    • Inconsistent examples are more useful

• Exact analyses of *optimal teaching errors* and *optimally incremental teachabilities* for concept classes of
  – $M_n^+$: Monotone monomials
  – $M_n'$: Monomials without the empty concept
  – $M_n$: Monomials
Our model

• Restriction: # of examples \( \leq k \)
  – Phone-a-friend lifeline: 30 sec.
  – This presentation: 25 min.
  – Lectures in our univ.: 90 min.

Millionaire 2.0 (two correct choices)
Concept class (available answers)
\( C = \{\{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \{C,D\}\} \)

Target concept (correct answer)
\( c = \{A, B\} \)

Learner (challenger)
Teacher (friend)
Our model

- Restriction: # of examples $\leq k$
  - Phone-a-friend lifeline: 30 sec.
  - This presentation: 25 min.
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Millionaire 2.0 (two correct choices)
Concept class (available answers)
$C = \{\{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \{C,D\}\}$

"The question is which two are ..."
(He used 29 sec.)

Target concept (correct answer)
c = \{A, B\}

Learner (challenger)
Teaching
Teacher (friend)
Our model

• Restriction: # of examples $\leq k$
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  – Lectures in our univ.: 90 min.

Teaching

Millionaire 2.0 (two correct choices)
Concept class (available answers)
$\mathcal{C} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$

“The question is which two are …”
(He used 29 sec.)

Target concept (correct answer)
$c = \{A, B\}$

“A is True. B is“

Learner (challenger)

S = \{(A, True)\}

Teacher (friend)
Our model

- Restriction: \# of examples \( \leq k \)
  - Phone-a-friend lifeline: 30 sec.
  - This presentation: 25 min.
  - Lectures in our univ.: 90 min.

Millionaire 2.0 (two correct choices)
Concept class (available answers)
\( C = \{\{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \{C,D\}\} \)

Target concept (correct answer)
\( c = \{A\} \)

“\( A \) is True. \( B \) is”
(He used 29 sec.)

Learner (challenger)
Teaching
Teacher (friend)
Optimal Teaching Error

**Definition**

\[ \text{OptTErr}_k(c, C) := \min_{S: |S| \leq k} \max_{h \in \text{CONS}(S, C)} \text{Err}(c, h) \]

\[ \text{OptTSets}_k(c, C) := \arg \min_{S: |S| \leq k} \max_{h \in \text{CONS}(S, C)} \text{Err}(c, h) \]

\[ \text{Err}(c, h) := \frac{|c \Delta h|}{|X|} = \frac{|c \cup h - c \cap h|}{|\{A, B, C, D\}|} \]

**Worst case error**

\[ \text{k-optimal teaching sets} \] achieving the optimal teaching error

Millionaire 2.0 (two correct choices)

\[ C = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \]

\[ c = \{A, B\} \]

\[ \text{OptTSets}_1(c, C) = \{(A, True), (B, True)\} \]
\[ \{(C, False), (D, False)\} \]

\[ \text{OptTSets}_2(c, C) = \text{MinTSets}(c, C) \]
\[ = \{(A, True), (B, True)\}, \{(C, False), (D, False)\} \]
Optimal Teaching Error

**Definition**

\[
\text{OptTErr}_k(c, C) := \min_{S:|S|\leq k} \max_{h\in \text{CONS}(S,C)} \text{Err}(c, h)
\]

\(k\)-optimal teaching sets achieving the optimal teaching error

\[
\text{OptTSets}_k(c, C) := \arg \min_{S:|S|\leq k} \max_{h\in \text{CONS}(S,C)} \text{Err}(c, h)
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<table>
<thead>
<tr>
<th>(h)</th>
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\(\text{Err}(c, \{A,C\}) = 2/4\) (two correct choices)

\[c = \{A, B\}\]

\[\text{OptTSets}_1(c, C) = \{(A, \text{True}), (B, \text{True})\}
\{(C, \text{False}), (D, \text{False})\}\]

\[\text{OptTSets}_2(c, C) = \text{MinTSets}(c, C)
\]

\[
\text{MinTSets}(c, C) := \{ (c, \text{True}), (b, \text{True}) , (c, \text{False}), (d, \text{False}) \}
\]

\[
\text{Err}(c, h) := \left| \frac{|c \Delta h|}{|X|} \right| = \left| \frac{|c \cup h - c \cap h|}{|\{A, B, C, D\}|} \right|
\]

Optimal Teaching Error

**Definition**

\[
OptTErr_k(c, C) := \min_{S:|S|\leq k} \max_{h \in CONS(S, C)} Err(c, h)
\]

**Worst case error**

\[
OptTSets_k(c, C) := \arg\min_{S:|S|\leq k} \max_{h \in CONS(S, C)} Err(c, h)
\]

**k-optimal teaching sets** achieving the optimal teaching error

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\[c = \{A, B\}\]

Millionaire 2.0 (two correct choices)

\[C = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}\]

\[c = \{A, B\}\]

\[\text{OptTSets}_1(c, C) = \{(A, True), (B, True)\}, \{(C, False), (D, False)\}\]

\[\text{OptTSets}_2(c, C) = \text{MinTSets}(c, C) = \{(A, True), (B, True), (C, False), (D, False)\}\]

\[Err(c, \{C,D\}) = 4/4\]
# Optimal Teaching Error

**Definition**

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OptTErr_k(c, C) := \min_{S:|S| \leq k} \max_{h \in CONS(S, C)} Err(c, h)
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OptTSets_k(c, C) := \arg \min_{S:|S| \leq k} \max_{h \in CONS(S, C)} Err(c, h)
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---

**Worst case error**

\[
Err(c, h) := \frac{|c \Delta h|}{|X|} = \frac{|c \cup h - c \cap h|}{|\{A, B, C, D\}|}
\]

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## k-optimal teaching sets achieving the optimal teaching error

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- **Worst case error = 2/4** if teaching \((A, True)\)
- \(\) (It is optimal)

\[
OptTSets_1(c, C) = \{(A, True)\}, \{(B, True)\}, \{(C, False)\}, \{(D, False)\}
\]

\[
OptTSets_2(c, C) = \text{MinTSets}(c, C) = \{(A, True), (B, True)\}, \{(C, False), (D, False)\}
\]
# Optimal Teaching Error

**Definition**

\[
OptTErr_k (c, C) := \min_{S: |S| \leq k} \max_{h \in CONS(S,C)} Err(c, h)
\]

**Worst case error**

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OptTSets_k (c, C) := \arg \min_{S: |S| \leq k} \max_{h \in CONS(S,C)} Err(c, h)
\]

**k-optimal teaching sets** achieving the optimal teaching error

### Table: Teaching Examples

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**Millionaire 2.0 (two correct choices)**

\[ C = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \]

**c = \{A, B\}**

**Worst case error = 4/4**

if teaching (A, False)

(It is NOT optimal)

\[ Err(c, h) := \frac{|c \Delta h|}{|X|} = \frac{|c \cup h - c \cap h|}{|\{A, B, C, D\}|} \]

C = \{(A, True), (B, True), (C, False), (D, False)\} 

\[ OptTSets_2(c, C) = \text{MinTSets}(c, C) = \{\{(A, True), (B, True)\},\{(C, False), (D, False)\}\} \]
Features of our model (1/2)

Theorem

\[
\exists \mathbf{C}, \exists c \in \mathbf{C}, \exists k > 0, \\
\forall S \in \text{MinTSets}(c, \mathbf{C}), \ \forall S' \subseteq S, \ S' \notin \text{OptTSets}_k(c, \mathbf{C}),
\]

– Special teaching strategies are necessary
  • A subset of a minimum teaching set is not always optimal

(Proof)

<table>
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<tr>
<td>c =</td>
<td>h</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
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\[
\mathbf{C} = \{\{A, B, C\}, \{A, C\}, \{A\}, \{B, C\}, \{B\}\}
\]
\[
c = \{A, B, C\}
\]
\[
k = 1
\]
\[
\text{MinTSets}(c, \mathbf{C}) = \{(A, \text{True}), (B, \text{True})\}
\]
\[
\text{OptTSets}_1(c, \mathbf{C}) = \{(C, \text{True})\}
\]
Features of our model (1/2)

\[ \exists \mathbf{C}, \exists c \in \mathbf{C}, \exists k > 0, \]
\[ \forall S \in \text{MinTSets}(c, \mathbf{C}), \forall S' \subseteq S, S' \notin \text{OptTSets}_k(c, \mathbf{C}), \]

– Special teaching strategies are necessary

• A subset of a minimum teaching set is not always optimal

(Proof)

\[
\begin{array}{|c|c|c|c|}
\hline
h & A & B & C & \text{Err}(c, h) \\
\hline
\{A, B, C\} & T & T & T & 0/3 \\
\{A, C\} & T & F & T & 1/3 \\
\{A\} & T & F & F & 2/3 \\
\{B, C\} & F & T & T & 1/3 \\
\{B\} & F & T & F & 2/3 \\
\hline
\end{array}
\]

\[ c = \{\{A, B, C\}, \{A, C\}, \{A\}, \{B, C\}, \{B\}\} \]
\[ c = \{A, B, C\} \]
\[ k = 1 \]

\[ \text{MinTSets}(c, \mathbf{C}) = \{ (A, True), (B, True) \} \]
\[ \text{OptTSets}_1(c, \mathbf{C}) = \{ (C, True) \} \]
Features of our model (1/2)

- Special teaching strategies are necessary
  - A subset of a minimum teaching set is not always optimal

(Proof)

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C = \{\{A, B, C\}, \{A, C\},
\{A\}, \{B, C\}, \{B\}\}

MinTSets(c, C) = \{(A, True), (B, True)\}

OptTSets_1(c, C) = \{(C, True)\}

Worst case error = 2/3

\[ \exists C, \exists c \in C, \exists k > 0, \]
\[ \forall S \in \text{MinTSets}(c, C), \forall S' \subseteq S, S' \notin \text{OptTSets}_k(c, C), \]
Features of our model (1/2)

**Theorem**

\[
\exists \mathcal{C}, \ \exists c \in \mathcal{C}, \ \exists k > 0, \\
\forall S \in \text{MinTSets}(c, \mathcal{C}), \ \forall S' \subseteq S, \ S' \notin \text{OptTSets}_k(c, \mathcal{C}),
\]

- Special teaching strategies are necessary
  - A subset of a minimum teaching set is not always optimal

(Proof)

<table>
<thead>
<tr>
<th>h</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Err(c, h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{A,B,C}</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>0/3</td>
</tr>
<tr>
<td>{A,C}</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>1/3</td>
</tr>
<tr>
<td>{A}</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>2/3</td>
</tr>
<tr>
<td>{B,C}</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>1/3</td>
</tr>
<tr>
<td>{B}</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>2/3</td>
</tr>
</tbody>
</table>

**c =** \{A, B, C\}, \{A, C\}, \{A\}, \{B, C\}, \{B\}

\[c = \{A, B, C\}\]

\[k = 1\]

Worst case error = 2/3

\[\text{MinTSets}(c, \mathcal{C}) = \{(A, \text{True}), (B, \text{True})\}\]

\[\text{OptTSets}_1(c, \mathcal{C}) = \{(C, \text{True})\}\]
Features of our model (1/2)

– Special teaching strategies are necessary

• A subset of a minimum teaching set is not always optimal

\[
\exists C, \exists c \in C, \exists k > 0, \forall S \in MinTSets(c, C), \forall S' \subseteq S, S' \notin OptTSets_k(c, C),
\]

(Proof)

\[
\begin{array}{|c|c|c|c|}
\hline
h & A & B & C & Err(c, h) \\
\hline
\{A, B, C\} & T & T & T & 0/3 \\
\{A, C\} & T & F & T & 1/3 \\
\{A\} & T & F & F & 2/3 \\
\{B, C\} & F & T & T & 1/3 \\
\{B\} & F & T & F & 2/3 \\
\hline
\end{array}
\]

\[c = \{A, B, C\}, \{A, C\}, \{A\}, \{B, C\}, \{B\}\]

\[c = \{A, B, C\}\]

Worst case error = 1/3 (optimal)

\[\text{MinTSets}(c, C) = \{(A, \text{True}), (B, \text{True})\}\]

\[\text{OptTSets}_1(c, C) = \{(C, \text{True})\}\]
Features of our model (2/2)

Theorem

\[ \exists C, \exists c \in C, \exists k > 0, \]
\[ \forall S \in \text{OptTSets}_k(c, C), \ c \not\in \text{CONS}(S, C) \]

– Smart teachers dare to **tell a lie**

* A k-optimal teaching set can be **inconsistent** with c

---

<table>
<thead>
<tr>
<th>h</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>Err(c, h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{A,B,C,D,E}</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>0/5</td>
</tr>
<tr>
<td>{B,C,D,E}</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>1/5</td>
</tr>
<tr>
<td>{A,B}</td>
<td>T</td>
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<td>F</td>
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</tr>
<tr>
<td>{A,C}</td>
<td>T</td>
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<td>F</td>
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</tr>
<tr>
<td>{A,D}</td>
<td>T</td>
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<tr>
<td>{A,E}</td>
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<td>F</td>
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<td>3/5</td>
</tr>
</tbody>
</table>

(Proof)

\[ c = \{\{A, B, C, D, E\}, \{B, C, D, E\}, \{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}\} \]
\[ k = 1 \]

\[ \text{OptTSets}_1(c, C) = \{ \{(A, \text{False})\} \} \]
\[ \text{CONS}(\{(A, \text{False})\}, C) = \{ \{B, C, D, E\}\} \]
Features of our model (2/2)

- Smart teachers dare to tell a lie
  - A k-optimal teaching set can be inconsistent with c

<table>
<thead>
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</tr>
<tr>
<td>{B,C,D,E}</td>
<td>F</td>
<td>T</td>
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<td>T</td>
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</tr>
<tr>
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<td>{A,C}</td>
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<tr>
<td>{A,E}</td>
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<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>3/5</td>
</tr>
</tbody>
</table>

OptTSets$_1$(c, C) = \{ (A, False) \}
CONS( \{(A, False)\}, C) = \{ {B, C, D, E} \}

\[ \exists C, \exists c \in C, \exists k > 0, \]
\[ \forall S \in \text{OptTSets}_k(c, C), \ c \not\in \text{CONS}(S, C) \]

(Proof)

Worst case error = 1/5 (optimal) although (A, False) is a lie

(Proof)
Features of our model (2/2)

Theorem

\[ \forall S \in \text{OptTSets}_k(c, \mathcal{C}), \ c \not\in \text{CONS}(S, \mathcal{C}) \]

– Smart teachers dare to **tell a lie**

• A k-optimal teaching set can be **inconsistent** with \( c \)

---

### Proof

<table>
<thead>
<tr>
<th>( h )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>( \text{Err}(c, h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{A,B,C,D,E}</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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</tr>
<tr>
<td>{B,C,D,E}</td>
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<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>3/5</td>
</tr>
</tbody>
</table>

\( c = \{A, B, C, D, E\}, \{B, C, D, E\}, \{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}\)

Worst case error = 3/5

If teaching the truth (A, True)

\( \text{OptTSets}_1(c, \mathcal{C}) = \{ \{(A, \text{False})\} \} \)

\( \text{CONS}( \{(A, \text{False})\}, \mathcal{C} ) = \{ \{B,C,D,E\} \} \)
Optimally Incremental Teachability

**Definition**

$c$ is optimally incrementally teachable w.r.t. $\mathcal{C}$

$\triangleq \text{def}
\exists \langle z_1, \ldots, z_{TD(c, \mathcal{C})} \rangle, \forall k \in [1, TD(c, \mathcal{C})],$

$\{z_1, \ldots, z_k\} \in \text{OptTSets}_k(c, \mathcal{C})$

– Optimal teaching strategies independent of $k$

**Fact**

$c$ of Millionaire 2.0 is opt. inc. teachable w.r.t. $\mathcal{C}$

Millionaire 2.0 (two correct choices)

$\mathcal{C} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$

$c = \{A, B\}$

$$\begin{align*}
\text{OptTSets}_2(c, \mathcal{C}) &= \text{MinTSets}(c, \mathcal{C}) \\
&= \{(A, True), (B, True)\}, \{(C, False), (D, False)\}
\end{align*}$$

$$\begin{align*}
\text{OptTSets}_1(c, \mathcal{C}) &= \{(A, True), (B, True), (C, False), (D, False)\}
\end{align*}$$

1-opt.  2-opt.

$\langle (A, True), (B, True) \rangle$
Concept classes $M_n^+, M_n', M_n$

- $M_n$: Monomials
  - A concept is a set of $x \in \{0,1\}^n$ satisfying a monomial
    - Ex.) Concepts for monomials on 3 variables
      - $v_1 : \{100, 101, 110, 111\}$ ← Monotone monomial
      - $\overline{v_1}v_2\overline{v_3} : \{010\}$
      - $v_1\overline{v_1} : \phi$ (Empty concept)

- $M_n'$: Monomials w/o the empty concept
  - $M_n' := M_n - \{\phi\}$

- $M_n^+$: Monotone monomials
  - Monomials consisting of only positive literals
Opt. inc. teachability of $M_n^+$

**Theorem**

$M_n^+$ is opt. inc. teachable

(Proof sketch)

The condition for $c$ is satisfied by $\langle z_1, z_2, \ldots \rangle$

$$z_i := \begin{cases} (1^{i-1}01^{n-i}, False) & (i \leq \ell) \\ (1^\ell \bar{0}^{n-\ell}, True) & (i > \ell) \end{cases}$$

$\ell$: # of variables of the target monomial

Ex.) $c \in M_3^+$: $v_1v_2$

<table>
<thead>
<tr>
<th>$h \in M_3^+$</th>
<th>$\text{Err}(c, h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(All true)</td>
<td>$6/2^3$</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$2/2^3$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$2/2^3$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$4/2^3$</td>
</tr>
<tr>
<td>$v_1v_2$</td>
<td>$0/2^3$</td>
</tr>
<tr>
<td>$v_2v_3$</td>
<td>$2/2^3$</td>
</tr>
<tr>
<td>$v_1v_3$</td>
<td>$2/2^3$</td>
</tr>
<tr>
<td>$v_1v_2v_3$</td>
<td>$1/2^3$</td>
</tr>
</tbody>
</table>

Minimum teaching set

$z_1 = (0 \ 1 \ 1, \ False)$
$z_2 = (1 \ 0 \ 1, \ False)$
$z_3 = (1 \ 1 \ 0, \ True)$

c $\rightarrow$ Optimal order
**Opt. inc. teachability of $M_{n^+}$**

**Theorem**

$M_{n^+}$ is opt. inc. teachable

(Proof sketch)

The condition for $c$ is satisfied by $\langle z_1, z_2, \ldots \rangle$

$$z_i := \begin{cases} (1^{i-1}01^{n-i}, False) & (i \leq \ell) \\ (1^{\ell}0^{n-\ell}, True) & (i > \ell) \end{cases}$$

$\ell$: # of variables of the target monomial

Ex.) $c \in M_{3^+}$: $v_1 v_2$

- $z_1 = (011, False)$
- $z_2 = (101, False)$
- $z_3 = (110, True)$

Optimal order

Minimum teaching set

<table>
<thead>
<tr>
<th>$h \in M_{3^+}$</th>
<th>$Err(c, h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(All true)</td>
<td>6/2³</td>
</tr>
<tr>
<td>$v_1$</td>
<td>2/2³</td>
</tr>
<tr>
<td>$v_2$</td>
<td>2/2³</td>
</tr>
<tr>
<td>$v_3$</td>
<td>4/2³</td>
</tr>
<tr>
<td>$v_1 v_2$</td>
<td>0/2³</td>
</tr>
<tr>
<td>$v_2 v_3$</td>
<td>2/2³</td>
</tr>
<tr>
<td>$v_1 v_3$</td>
<td>2/2³</td>
</tr>
<tr>
<td>$v_1 v_2 v_3$</td>
<td>1/2³</td>
</tr>
</tbody>
</table>

by $z_1$

by $z_1$

by $z_1$

by $z_1$

by $z_1$
Opt. inc. teachability of $M_n^+$

**Theorem**

$M_n^+$ is opt. inc. teachable

(Proof sketch)

The condition for $c$ is satisfied by $\langle z_1, z_2, \ldots \rangle$

$$z_i := \begin{cases} (1^{i-1}01^{n-i}, False) & (i \leq \ell) \\ (1^\ell0^{n-\ell}, True) & (i > \ell) \end{cases}$$

$\ell$: # of variables of the target monomial

Ex.) $c \in M_3^+$:

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<tr>
<td>(All true)</td>
<td>6/2^3</td>
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<tr>
<td>$v_1$</td>
<td>2/2^3</td>
</tr>
<tr>
<td>by $z_1$</td>
<td></td>
</tr>
<tr>
<td>$v_2$</td>
<td>2/2^3</td>
</tr>
<tr>
<td>by $z_1$</td>
<td></td>
</tr>
<tr>
<td>$v_3$</td>
<td>4/2^3</td>
</tr>
<tr>
<td>by $z_1$</td>
<td></td>
</tr>
<tr>
<td>$v_1v_2$</td>
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<tr>
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<td></td>
</tr>
<tr>
<td>$v_2v_3$</td>
<td>2/2^3</td>
</tr>
<tr>
<td>by $z_1$</td>
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</tr>
<tr>
<td>$v_1v_3$</td>
<td>2/2^3</td>
</tr>
<tr>
<td>by $z_2$</td>
<td></td>
</tr>
<tr>
<td>$v_1v_2v_3$</td>
<td>1/2^3</td>
</tr>
<tr>
<td>by $z_1$</td>
<td></td>
</tr>
</tbody>
</table>

Minimum teaching set

$z_1 = (0\ 1\ 1, \ False)$

$z_2 = (1\ 0\ 1, \ False)$

$z_3 = (1\ 1\ 0, \ True)$

$\iff$

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Opt. inc. teachability of $M_{n^+}$

(Proof sketch)

The condition for $c$ is satisfied by $\langle z_1, z_2, \ldots \rangle$

$$z_i := \begin{cases} (1^{i-1} 0^1, \text{False}) & (i \leq \ell) \\ (1^\ell 0^{n-\ell}, \text{True}) & (i > \ell) \end{cases}$$

$\ell$: # of variables of the target monomial

Ex.) $c \in M_{3^+}$:

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<thead>
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<tr>
<td>$(\text{All true})$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$v_1$</td>
<td>$2/2^3$</td>
<td>by $z_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_2$</td>
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<td>by $z_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_3$</td>
<td>$4/2^3$</td>
<td>by $z_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_1v_2$</td>
<td>$0/2^3$</td>
<td>by $z_1$</td>
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<td></td>
</tr>
<tr>
<td>$v_2v_3$</td>
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<td>by $z_3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Opt. inc. teachability of $M_n'$

$M_n'$ is **not** opt. inc. teachable

(Proof sketch)

$z_{opt} := (0^n, \text{False})$ is a special teaching strategy when $k=1$

<table>
<thead>
<tr>
<th>$h \in M_3'$</th>
<th>$\text{Err}(c, h)$</th>
<th>$h \in M_3'$</th>
<th>$\text{Err}(c, h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(All true)</td>
<td>$6/2^3$</td>
<td>$\overline{v}_1$</td>
<td>$6/2^3$</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$2/2^3$</td>
<td>$\overline{v}_2$</td>
<td>$6/2^3$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$2/2^3$</td>
<td>$\overline{v}_3$</td>
<td>$4/2^3$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$4/2^3$</td>
<td>$\overline{v}_1v_2$</td>
<td>$4/2^3$</td>
</tr>
<tr>
<td>$v_1v_2$</td>
<td>$0/2^3$</td>
<td>$v_1\overline{v}_2$</td>
<td>$4/2^3$</td>
</tr>
<tr>
<td>$v_2v_3$</td>
<td>$2/2^3$</td>
<td>$\overline{v}_1v_2$</td>
<td>$4/2^3$</td>
</tr>
<tr>
<td>$v_1v_3$</td>
<td>$2/2^3$</td>
<td>$\overline{v}_2v_3$</td>
<td>$4/2^3$</td>
</tr>
<tr>
<td>$v_1v_2v_3$</td>
<td>$1/2^3$</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Ex.) $c \in M_3'$:

$z_{opt} = (0\ 0\ 0, \text{False})$

$z_0 = (1\ 1\ 1, \text{True})$
$z_1 = (0\ 1\ 1, \text{False})$
$z_2 = (1\ 0\ 1, \text{False})$
$z_3 = (1\ 1\ 0, \text{True})$

1-optimal teaching set

Minimum teaching set

Not all negated

Not subset

$z_{opt}$ is not subset of $M_n'$
Opt. inc. teachability of $M_n'$

**Theorem**

$M_n'$ is **not** opt. inc. teachable

(Proof sketch)

$z_{\text{opt}} := (0^n, \text{False})$ is a special teaching strategy when $k=1$

**Ex.)** $c \in M_3'$:

$\begin{align*}
    v_1v_2 & \quad 4/2^3 \\
    z_0 &= (1 1 1, \text{True}) \\
    z_1 &= (0 1 1, \text{False}) \\
    z_2 &= (1 0 1, \text{False}) \\
    z_3 &= (1 1 0, \text{True}) \\
\end{align*}$

$z_{\text{opt}} = (0 0 0, \text{False})$

1-optimal teaching set

Not negated

For $M_n^+$

Minimum teaching set

$$
\begin{array}{|c|c|c|}
\hline
h \in M_3' & \text{Err}(c, h) & h \in M_3' & \text{Err}(c, h) \\
\hline
(\text{All true}) & 6/2^3 & \overline{v}_1 & 6/2^3 \\
\overline{v}_1 & 2/2^3 & \overline{v}_2 & 6/2^3 \\
\overline{v}_2 & 2/2^3 & \overline{v}_3 & 4/2^3 \\
\overline{v}_3 & 4/2^3 & \overline{v}_1\overline{v}_2 & 4/2^3 \\
v_1v_2 & 0/2^3 & v_1\overline{v}_2 & 4/2^3 \\
v_2v_3 & 2/2^3 & v_1\overline{v}_2 & 4/2^3 \\
v_1v_3 & 2/2^3 & \overline{v}_2v_3 & 4/2^3 \\
v_1v_2v_3 & 1/2^3 & \ldots & \ldots \\
\hline
\end{array}
$$

**Minimum teaching set**

Montreal, Canada. 21 June.
Opt. inc. teachability of $M_n'$

**Theorem**

$M_n'$ is **not** opt. inc. teachable

(Proof sketch)

$z_{opt} := (0^n, \text{False})$ is a special teaching strategy when $k=1$

Ex.) $c \in M_3': v_1v_2$

$z_{opt} = (0 0 0, \text{False})$

$z_0 = (1 1 1, \text{True})$

$z_1 = (0 1 1, \text{False})$

$z_2 = (1 0 1, \text{False})$

$z_3 = (1 1 0, \text{True})$

\[ \begin{array}{c|c|c|c|c}
 h & \text{Err}(c, h) & h & \text{Err}(c, h) \\
 \hline
 (\text{All true}) & 6/2^3 & \overline{v_1} & 6/2^3 \\
 v_1 & 2/2^3 & \overline{v_2} & 6/2^3 \\
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 v_1v_2v_3 & 1/2^3 & \ldots & \ldots \\
\end{array} \]

Not negated

1-optimal

Can’t exclude (All true)

Not subset

Minimum teaching set
Opt. inc. teachability of $M_n'$

**Theorem**

$M_n'$ is **not** opt. inc. teachable

(Proof sketch)

$z_{\text{opt}} := (0^n, \text{False})$ is a special teaching strategy when $k=1$

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<td>(All true)</td>
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</tr>
<tr>
<td>$v_1$</td>
<td>$2/2^3$</td>
<td>$\overline{v}_2$</td>
<td>$6/2^3$</td>
</tr>
<tr>
<td>$v_2$</td>
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Ex.) $c \in M_3'$: $v_1v_2$

$z_{\text{opt}} = (0 0 0, \text{False})$

- Not negated
- 1-optimal
- Not subset
- Can’t exclude (All true)
- Can’t exclude $\overline{v}_2$

$z_0 = (1 1 1, \text{True})$
$z_1 = (0 1 1, \text{False})$
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Minimum teaching set
Opt. inc. teachability of $M_n'$

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Minimum teaching set

Montreal, Canada. 21 June.
Opt. inc. teachability of $M_n'$

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Ex.) $c \in M_3'$: $v_1v_2$

1-optimal

Not negated

$z_{\text{opt}} = (0 \ 0 \ 0, \text{False})$

Minimum teaching set

Montreal, Canada. 21 June.
Opt. inc. teachability of $M_n'$

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Minimum teaching set
Interesting result

∀k ∈ [4, 2^{n-1} – 1],
∀S ∈ OptTSets_k(φ, M_n),  φ ∉ CONS(S, M_n)

– Teachers must tell a lie to **optimally** teach φ in M_n

(Proof sketch when k>n)

TD(c', M_n) = n+1

S ∈ MinTSets(c', M_n) is k-optimal for c

However, S is inconsistent with c

<table>
<thead>
<tr>
<th>h ∈ M_n</th>
<th>Err(c, h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ</td>
<td>0/2^n</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>v_1 v_2... v_n</td>
<td>1/2^n</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Theorem ([Goldman and Kearns 1995])

TD(c, M_n) = min{ℓ + 2, n + 1}
Interesting result – Teachers must tell a lie to optimally teach $\phi$ in $M_n$

Proof sketch when $k > n$)

$TD(c', M_n) = n + 1$

$S \in \text{MinTSets}(c', M_n)$ is $k$-optimal for $c$

However, $S$ is inconsistent with $c$

<table>
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<tr>
<th>$h \in M_n$</th>
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<tr>
<td>$\phi$</td>
<td>$0/2^n$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$v_1 v_2 \ldots v_n$</td>
<td>$1/2^n$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Theorem

\[ \exists C, \exists c \in C, \exists k > 0, \]
\[ \forall S \in OptTSets_k(c, C), \ c \not\in CONS(S, C) \]

Theorem

\[ \forall k \in [4, 2^{n-1} - 1], \]
\[ \forall S \in OptTSets_k(\phi, M_n), \ \phi \not\in CONS(S, M_n) \]

– Teachers must tell a lie to optimally teach \( \phi \) in \( M_n \)

(Proof sketch when \( k > n \))

(Collected by M_n : \( v_1 v_2 \ldots v_n \)
\[ TD(c', M_n) = n+1 \]

\[ S \in MinTSets(c', M_n) \text{ is } k\text{-optimal for } c \]

However, \( S \) is inconsistent with \( c \)

\[ TD(c', M_n) = \min \{ \ell + 2, n + 1 \} \]

\[ h \in M_n \quad \text{Err}(c, h) \]
\[ \phi \quad 0/2^n \]
\[ \ldots \quad \ldots \]
\[ v_1 v_2 \ldots v_n \quad 1/2^n \]
\[ \ldots \quad \ldots \]
## Summary

<table>
<thead>
<tr>
<th></th>
<th>$M^+_n$</th>
<th>$M'_n$</th>
<th>$M_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Teaching Dim., TD(C)</strong></td>
<td>$n$</td>
<td>$n+1$</td>
<td>$2^n$</td>
</tr>
<tr>
<td>[Goldman and Kearns 1991]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Teachability</strong></td>
<td>True</td>
<td>True</td>
<td>False</td>
</tr>
<tr>
<td>[Shinohara and Miyano 1991]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Opt. Teaching Error OptTE_k(C)</strong></td>
<td>$\frac{2^{n-k} - 1}{2^n}$</td>
<td>$\frac{2^{n-k+1} - 1}{2^n}$</td>
<td>$\begin{cases} \frac{2^{n-k+1} - 1}{2^n} &amp; (k \leq 2) \ \frac{2^{n-k+1}}{2^n} &amp; (2 &lt; k \leq n) \ \frac{1}{2^n} &amp; (n &lt; k &lt; 2^n) \end{cases}$</td>
</tr>
<tr>
<td><strong>Opt. Inc. Teachability</strong></td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
</tbody>
</table>

**Our results**

- Different boundary
- Quite small

---

*COLT2009. Montreal, Canada. 21 June.*
Thank you for your attention!